Evolution of the unidirectional electromagnetic pulses in an anisotropic two-level medium

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We find a general integrable system of reduced Maxwell-Bloch equations describing the interaction of a two-component electromagnetic field with the dipole transition of a two-level medium. The anisotropy of the anisotropic dipole momentum of the transition as a permanent dipole momentum is taken into account. A solution of the model is found for a particular case by using the inverse scattering transform. The method is based on a solution of the Riemann-Hilbert problem taking into account the symmetry properties of the corresponding fundamental solutions.

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I. INTRODUCTION

The generation and evolution of so-called few-cycle pulses with duration comparable to the oscillation period $2\pi/\omega_0$, where ω_0 is a transition frequency, are the subject of ongoing research motivated by applications in various branches of physics (e.g., see the review in [1]). It was noted in various publications that theoretical methods developed for analyzing femtosecond optical-pulse propagation do not meet the requirements dictated by experiment. In particular, the pulse dynamics is analyzed by using the slowly varying envelope approximation with small corrections allowing for deviations from this approximation [1]. In [2], numerical methods were applied to solve the semiclassical Maxwell-Bloch (MB) equations [3]. In those and other studies, the self-induced transparency was analyzed for linearly polarized pulses interacting with a two-level system. However, numerical methods may not provide a sufficiently detailed and reliable characterization of the dynamics governed by complicated systems of equations, such as the Maxwell-Bloch system describing femtosecond optical-pulse propagation. Detailed analytical information can be obtained by using the inverse scattering transform (IST) for a solution of integrable models [4]. The evolution of quasimonochromatic pulses propagating in a two-level medium has been studied in detail in the framework of integrable Maxwell-Bloch equations by different authors [3,5–9].

The applicability of the two-level model to such pulses requires that the resonant frequency be well separated from other frequencies [10]. For the quasimonochromatic pulses this condition is satisfied as usual. For few-cycle pulses such an applicability condition becomes less realistic; however, it is improved if the dipole moment corresponding to the resonant transition is larger than those of the nearest transitions [11].

The period of a light wave with a wavelength of 780 nm is about 2.6 fs. Currently, a Ti:sapphire laser can be used to generate 7.5-fs pulses [12] and 4.5-fs pulses with the use of a fiber-optic pulse compressor [13]. These pulse durations are only a few times longer than the oscillation period. Therefore, their dynamics cannot be analyzed by using solutions obtained in models with slowly varying amplitudes and phases as zeroth approximations. These pulses are not sufficiently short to justify the use of the ultrashort-pulse approximation, which is applicable if $\tau_p \ll 2\pi/\omega_0$ (e.g., see [14]) and reduce the original Maxwell-Bloch system to simpler equations. Note that the latter approximation is not realistic in the optical frequency range. For this reason, simplifying approximations other than the condition $\tau_p \ll 2\pi/\omega_0$ should be employed. In this paper, the unidirectional approximation is used, in combination with other conditions, to reduce the Maxwell-Bloch equations to an integrable form. Accordingly, appropriate localized solutions to these evolution equations are called unidirectional pulses (UDPs). The concept of a UDP, which is widely used in fluid dynamics, was applied to a solution of the self-induced transparency (SIT) equations in [15], where the exact integrable reduced Maxwell-Bloch (RMB) equations were solved for plane polarization of light pulses.

These integrable RMB equations [15] generalized in Refs. [16–20] have remarkable structural properties. For example, we demonstrated that in the case of integrable anisotropic RMB equations some symmetry properties require suitable modification of the inverse scattering transform technique [20]. Following Refs. [17,18] we consider the interaction between an electromagnetic field (EMF) propagating in the z direction and an atomic two-level system with a dipole transition $\Delta J=0, \Delta M=1$ with J and M denoting the total angular momentum and its z component, respectively.

In the model of consideration we assume an anisotropy with respect to the dipole moment projections of two-levels dipole transition and an anisotropy of the permanent dipole momentum (PDM). The integrable RMB equations describing the interaction of the plane polarization of light with nonzero PDM were first established and studied in Ref. [16]. Following the previous papers [15,17,18] we use the verified unidirectional approximation for the Maxwell equations to take them in appropriate reduced integrable form.

In the following section we present the most general integrable RMB equations describing the interaction of a two-component UDP propagated in a one-dimensional two-level medium with anisotropic dipole transition and PDM. Next we consider a particular case of the model which allows us to find a relatively simple solution of the model by using the IST.

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II. GENERAL INTEGRABLE MODEL

The most general Hamiltonian describing the interaction of a two-component electromagnetic field with a two-level lossless and dispersionless medium in the dipole approximation is the following:

$$\hat{H} = \hat{\sigma}_3 \hbar\omega_0 - [\hat{\sigma}_{11}(d_{zx}^{(1)}E'_x + d_{zy}^{(1)}E'_y) + \hat{\sigma}_{22}(d_{zx}^{(2)}E'_x + d_{zy}^{(2)}E'_y) + \hat{\sigma}_1(d_{xx}E'_x + d_{xy}E'_y) + \hat{\sigma}_2(d_{yx}E'_x + d_{yy}E'_y)], \tag{1}$$

where $\hat{\sigma}_i$, $i=1,2,3$, are the standard Pauli matrices $\hat{\sigma}_{11}=\mathrm{diag}(1,0)$ and $\hat{\sigma}_{22}=\mathrm{diag}(0,1)$, E'_x and E'_y are the transverse components of the polarization of the EMF, and ω_0 is the frequency of the two-level transition. $d_{zs}^{(l)}$ and $d_{zs}^{(l)}$, $l=1,2$, $s=x,y$, are the real components of the PDM, and d_{sp}, $s,p=x,y$, are the real components of dipole momentum of the dipole transition between the levels $l=1,2$. Hamiltonian (1) takes into account the anisotropy of both the dipole transition and the anisotropy of the PDM.

We consider here the anisotropy of the dipole transition of impurity molecules. Such an anisotropy may be accompanied by the anisotropy of a host medium. For instance, the medium may be a two-axis one [21]. However, we do not take into account the third component of the electromagnetic field E'_z which arises in two-axis media. One is able to estimate the value of E'_z using, for example, the results of Ref. [22]:

$$E'_z \sim \delta E'_x = \frac{\epsilon_\perp - \epsilon_\parallel}{\epsilon_\perp} E'_x. \tag{2}$$

Here $\epsilon_\perp$ and $\epsilon_\parallel$ are the transverse and longitudinal components of the dielectric susceptibility constant. In most known birefringent media we found $\delta < 1\%-3\%$ [21]. Meantime the anisotropy of the dipole momenta may be of the order of unity. Therefore we neglect the third component E'_z below even for two-axis media.

Transform the components of the EMF,

$$\begin{pmatrix} \delta_x E_x \\ \delta_y E_y \end{pmatrix} = \begin{pmatrix} d_{xx} & d_{xy} \\ d_{yx} & d_{yy} \end{pmatrix} \begin{pmatrix} E'_x \\ E'_y \end{pmatrix} \tag{3}$$

or

$$\begin{pmatrix} E'_x \\ E'_y \end{pmatrix} = \frac{1}{P_0} \begin{pmatrix} \delta_x d_{yy} & -\delta_y d_{xy} \\ -\delta_x d_{yx} & \delta_y d_{xx} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}, \tag{4}$$

and introduce the following components of the effective dipole momenta:

$$\delta_x = \sqrt{d_{xx}^2 + d_{xy}^2}, \quad \delta_y = \sqrt{d_{yx}^2 + d_{yy}^2},$$

$$p_x^{(l)} = \frac{\delta_x}{P_0}(d_{zx}^{(l)} d_{yy} - d_{zy}^{(l)} d_{yx}),$$

$$p_x^{(l)} = \frac{\delta_y}{P_0}(d_{zy}^{(l)} d_{xx} - d_{zx}^{(l)} d_{xy}). \tag{5}$$

Here $P_0 = d_{xx}d_{yy} - d_{xy}d_{yx} \neq 0$, $l=1,2$. The case of $P_0=0$ must be considered separately.

Finally we obtain the effective Hamiltonian

$$\hat{H} = \hat{\sigma}_3 \hbar\omega_0 - [\hat{\sigma}_{11}(p_x^{(1)}E_x + p_y^{(1)}E_y) + \hat{\sigma}_{22}(p_x^{(2)}E_x + p_y^{(2)}E_y) + \hat{\sigma}_1 \delta_x E_x + \hat{\sigma}_2 \delta_y E_y]. \tag{6}$$

Transform the components of the EMF to a dimensionless form $\mathcal{E}_x, \mathcal{E}_y$:

$$\mathbf{E} = (E_x, E_y) = E_0(\mathcal{E}_x, \mathcal{E}_y). \tag{7}$$

Here $G_0 = \frac{d_0 E_0}{\omega_0 \hbar}$ is a dimensionless constant and $d_0 = \sqrt{4\delta_x^2 + 4\delta_y^2 + (p_x^{(1)} - p_x^{(2)})^2 + (p_y^{(1)} - p_y^{(2)})^2}$. Assume $G_0=1$.

Introduce the dimensionless components of the dipole momentum and the PDM,

$$\mu_x = \frac{2\delta_x}{d_0}, \quad m_x = \frac{p_x^{(1)} - p_x^{(2)}}{d_0},$$

$$\mu_y = \frac{2\delta_y}{d_0}, \quad m_y = \frac{p_y^{(1)} - p_y^{(2)}}{d_0}, \tag{8}$$

and the Bloch-vector components

$$S_x = \frac{\rho_{12} + \rho_{21}}{2}, \quad S_y = \frac{\rho_{12} - \rho_{21}}{2i}, \quad S_z = \frac{\rho_{11} - \rho_{22}}{2}, \tag{9}$$

where ρ_{ij} are the elements of the density matrix of the two-level medium.

Then the Bloch equations

$$i\hbar \frac{\partial \hat{\rho}}{\partial t} = [\hat{H}, \hat{\rho}] \tag{10}$$

take the form

$$\frac{\partial S_x}{\partial \tau'} = (1 - m_x \mathcal{E}_x - m_y \mathcal{E}_y) S_y + \mu_y \mathcal{E}_y S_z, \tag{11}$$

$$\frac{\partial S_y}{\partial \tau'} = (m_x \mathcal{E}_x + m_y \mathcal{E}_y - 1) S_x - \mu_x \mathcal{E}_x S_z, \tag{12}$$

$$\frac{\partial S_z}{\partial \tau'} = \mu_x \mathcal{E}_x S_y - \mu_y \mathcal{E}_y S_x. \tag{13}$$

Here $\tau' = \omega_0 t$.

The Maxwell equations we reduce using the approximation of the UDP propagation; see, for instance, [15,17,18]. The approximation is based on the following considerations. Frequently, the density of active atoms or molecules in a real medium is sufficiently low to be set as a small parameter. In other words, the normalized density of two-level atoms has an order of magnitude of the derivatives $\partial_z + n/c\partial_t$ of the field polarization. Here c/n is the phase light velocity and n is a constant of the dielectric susceptibility of the medium. In other words, this means that

$$\frac{\partial \mathcal{E}_{x,y}}{\partial z} \approx -\frac{n}{c}\frac{\partial \mathcal{E}_{x,y}}{\partial t}, \tag{14}$$

with the required accuracy.

Accordingly, the contribution due to the counterpropagating wave can be neglected. The resulting system of equations describes UDP propagation with a group velocity comparable to the speed of light in the medium [15].

The Maxwell equations are

$$\frac{\partial^2 \mathcal{E}_x}{\partial z^2} - \frac{n^2}{c^2}\frac{\partial^2 \mathcal{E}_x}{\partial t^2} = \frac{4\pi d_x}{c^2}\frac{\partial^2 P_x}{\partial t^2}, \tag{15}$$

$$\frac{\partial^2 \mathcal{E}_y}{\partial z^2} - \frac{n^2}{c^2}\frac{\partial^2 \mathcal{E}_y}{\partial t^2} = \frac{4\pi d_y}{c^2}\frac{\partial^2 P_y}{\partial t^2}, \tag{16}$$

where components of the medium polarizations $P_{x,y}$ are determined by the standard formula

$$P_{x,y} = -N\,\mathrm{Tr}\left\{\hat{\rho}\frac{\partial \hat{H}}{\partial \mathcal{E}_{x,y}}\right\}. \tag{17}$$

N is the density of the medium.

Under the unidirectional approximation the Maxwell equations (15)–(17) taking into account Bloch equations (11)–(13) are reduced to the equations

$$\frac{\partial \mathcal{E}_x}{\partial \chi'} = R_z \mathcal{E}_y - \mu_x S_y, \tag{18}$$

$$\frac{\partial \mathcal{E}_y}{\partial \chi'} = -R_z \mathcal{E}_x + \mu_y S_x. \tag{19}$$

Here

$$\chi' = \left(z + \frac{c}{n}t\right)\frac{4\pi N d_0^2}{\omega_0 n c \hbar},$$

$$R_z = m_x \mu_y S_x + m_y \mu_x S_y - \mu_x \mu_y S_z. \tag{20}$$

Starting from the Lax pair structure found by the author in Ref. [17] for the system (11)–(13), (18), and (19) in the case of $m_x = m_y = 0$, it may be directly shown that the system (11)–(13), (18), and (19) for any real constants is the condition of compatibility of the linear systems

$$\partial_\tau \Phi = \begin{pmatrix} -i\left(\lambda^2 - \frac{r^2}{\lambda^2}\right) & e^{i\phi}\left(\lambda G + \frac{r}{\lambda}\bar{G}\right) \\ e^{-i\phi}\left(\lambda \bar{G} + \frac{r}{\lambda}G\right) & i\left(\lambda^2 - \frac{r^2}{\lambda^2}\right) \end{pmatrix}\Phi, \tag{21}$$

$$\partial_\chi \Phi = \frac{1}{\Omega}\begin{pmatrix} i\left(\lambda^2 - \frac{r^2}{\lambda^2}\right)R_z & e^{i\phi}\left(\lambda R + \frac{r}{\lambda}\bar{R}\right) \\ e^{-i\phi}\left(\lambda \bar{R} + \frac{r}{\lambda}R\right) & -i\left(\lambda^2 - \frac{r^2}{\lambda^2}\right)R_z \end{pmatrix}\Phi. \tag{22}$$

Here $\tau' = 2\tau$, $\chi' = 2\chi$, and

$$r = |\tilde{r}|, \quad \tilde{r} = r\, e^{2i\phi},$$

$$\mathrm{Re}\,\tilde{r} = \frac{\Omega_x - \Omega_y}{4}, \quad \mathrm{Im}\,\tilde{r} = -\frac{M_{xy}}{2D}, \tag{23}$$

$$G = \mathcal{E}\, e^{-i\phi}, \quad R = \mathcal{R}\, e^{-i\phi}, \tag{24}$$

$$\mathcal{E} = \sqrt{\mu_x \mu_y D}(\mathcal{E}_y + i\mathcal{E}_x) + q_0, \tag{25}$$

$$q_0 = -\frac{1}{\sqrt{\mu_x \mu_y D}}\left(\frac{m_y \mu_x}{\mu_y} + i\frac{m_x \mu_y}{\mu_x}\right), \tag{26}$$

$$\mathcal{R} = \frac{\sqrt{\mu_x \mu_y}}{\sqrt{D}}\Bigg\{[\mu_x M_{xy} - i\mu_y(1 + M_{yy})]S_x + [i\mu_y M_{xy} - \mu_x(1 + M_{xx})]S_y - \left(i\frac{\mu_y m_x}{\mu_x} + \frac{\mu_x m_y}{\mu_y}\right)S_z\Bigg\}, \tag{27}$$

$$\Omega = \lambda^2 + \frac{r^2}{\lambda^2} + \frac{\Omega_x + \Omega_y}{2}, \tag{28}$$

$$\Omega_x = \frac{\mu_x}{D\mu_y}(1 + M_{xx}), \quad \Omega_y = \frac{\mu_y}{D\mu_x}(1 + M_{yy}), \tag{29}$$

$$D = 1 + M_{xx} + M_{yy}, \quad M_{ij} = \frac{m_i m_j}{\mu_i \mu_j}, \quad i,j = x,y. \tag{30}$$

The spectral problem (21) after rescaling $\lambda \to \sqrt{r}\lambda$ coincides with that found by the author [17] up to phase multipliers $e^{\pm i\phi}$. These phase factors may be incorporated in the techniques of inverse scattering transform (ISTM) of Refs. [19,20].

III. A PARTICULAR CASE

Application of the ISTM to the Lax pair (21) and (22) even for zero asymptotics of the EMF and the simplest initial-boundary conditions leads to a cumbersome and complicated theory due to nonzero off-diagonal parts of matrices on the right-hand side (RHS) sides of Eqs. (21) and (22) at the boundaries. Therefore we consider here in the following a simple particular case of the common system (11)–(13), (18), and (19) which is determined by the relations

$$\mu_x^2 + m_x^2 = \mu_y^2, \tag{31}$$

$$m_y = 0. \tag{32}$$

This case nevertheless contains some qualitatively novel features of the physical model and solutions.

The anisotropy of the interaction of the dipole transition with the EMF may be governed by the strong anisotropy of the ion medium. Therefore conditions (31) and (32) may be satisfied by choosing a respective position of the polarization of the EMF with respect to axis of symmetry of the medium.

From (31) and (32) we found from (23)

$$r, \phi \to 0. \tag{33}$$

After the changes $\lambda \to -i\lambda$ and $\tau \to -\tau$, the system of equations (21) and (22) takes the form

$$\partial_\tau \Phi = \begin{pmatrix} -i\lambda^2 & \lambda E \\ -\lambda \bar{E} & i\lambda^2 \end{pmatrix}\Phi \equiv \mathbf{L}\Phi, \tag{34}$$

$$\partial_\chi \Phi = \frac{i}{\lambda^2 - 1} \begin{pmatrix} \lambda^2 R_z & \lambda R \\ \lambda \bar{R} & -\lambda^2 R_z \end{pmatrix} \Phi \equiv \mathbf{A}\Phi, \tag{35}$$

where R_z and R are determined by formulas (20), (27), and (28), respectively taking into account conditions (31) and (32) and

$$E = \frac{\sqrt{\mu_y^3}}{\sqrt{\mu_x}}(i\mathcal{E}_y - \mathcal{E}_x) + q, \tag{36}$$

$$q = \frac{m_x}{\sqrt{\mu_x \mu_y}}. \tag{37}$$

The linear system (34) is the Kaup-Newell spectral problem [23]. This problem was solved for zero density (i.e., $E(\tau)=0$, $\tau \to \pm\infty$ [23]) and for nonzero density (i.e., $E(\tau)=\mathrm{const} \neq 0$, $\tau \to \pm\infty$ [24]).

We consider the evolution of finite supported pulses of the EMF, i.e.,

$$\mathcal{E}_x, \mathcal{E}_y \to 0, \quad \tau \to \pm\infty, \tag{38}$$

$$S_x, S_y \to 0, \quad \tau \to \pm\infty. \tag{39}$$

The zero asymptotic conditions (38) differ from those considered in Refs. [24,25]. In the present paper the fields tend to zero, but the off-diagonal elements of the linear matrices **L** and **A** tend to nonzero constants as $\tau \to \pm\infty$. Due to this fact, obtained here, solutions for $\mathcal{E}_{x,y}$ differ from those obtained in Refs. [24,25].

A way to avoid the ambiguities arising due to the square-root form of the eigenvalues of the spectral problem analogous to (21) with nonzero density has been proposed in [26]. Following this way, introduce the spectral parameter ξ and its functions

$$\lambda(\xi) = \frac{1}{2}\left(\xi + \frac{q^2}{\xi}\right), \quad \Lambda(\xi) = \frac{1}{4}\left(\xi^2 - \frac{q^4}{\xi^2}\right). \tag{40}$$

Introduce the matrix-valued functions

$$\Phi'_- = (\phi', \tilde{\phi}'), \quad \Phi'_+ = (\tilde{\psi}', \psi'), \tag{41}$$

Here $\phi' = \phi'(\chi, \tau; \lambda)$ and $\tilde{\phi}' = \tilde{\phi}'(\chi, \tau; \lambda)$, are the columns. These matrix-functions possess the asymptotics

$$\Phi'_\pm(\tau;\xi) \to \mathbf{F}_\pm(\tau;\xi) = \begin{pmatrix} 1 & -iq\xi^{-1} \\ -iq\xi^{-1} & 1 \end{pmatrix} e^{-i\Lambda(\xi)\tau\hat{\sigma}_3},$$
$$\tau \to \pm\infty. \tag{42}$$

The Jost functions, which are fundamental solutions of the system (34), take the form

$$\Phi^- = (e^{(-i\alpha_- + i\alpha_0)\hat{\sigma}_3}\phi', \; e^{(i\alpha_- - i\alpha_0)\hat{\sigma}_3}\tilde{\phi}') := (\phi, \tilde{\phi}),$$

$$\Phi^+ = (e^{-i\alpha_+\hat{\sigma}_3}\tilde{\psi}', \; e^{i\alpha_+\hat{\sigma}_3}\psi') := (\tilde{\psi}, \psi). \tag{43}$$

Here $\alpha_\pm$ are the real functions of τ and χ having the asymptotics

$$\lim_{\tau \to -\infty} \alpha_-(\tau, \chi) = 0, \quad \lim_{\tau \to \infty} \alpha_+(\tau, \chi) = 0 \tag{44}$$

and α_0 is a real constant.

IV. SYMMETRY PROPERTIES

The completeness condition and symmetry property [see below Eq. (50)] yield the determination of the scattering matrix **T**:

$$\Phi^- = \Phi^+ \mathbf{T}, \quad \mathbf{T} = \begin{pmatrix} a(\xi) & -\overline{b(\bar{\xi})} \\ b(\xi) & \overline{a(\bar{\xi})} \end{pmatrix}. \tag{45}$$

Here $\xi \in \Gamma = \{\xi : \mathrm{Im}(\xi^2) = 0\}$.

Determine the group of the transform of the complex plane ξ, which includes the identical transform I and the elements acting as follows:

$$g_1(\xi) = \frac{q^2}{\bar{\xi}}, \quad g_2(\xi) = -\xi, \quad g_3(\xi) = -\frac{q^2}{\bar{\xi}}. \tag{46}$$

The transforms $\{I, u_{g_1}, u_{g_2}, u_{g_3}\}$ forming an Abelian group $\mathcal{S}$ of substitutions include the parity transform u_{g_2}, the substitution u_{g_2}, and the combined transform u_{g_3} and $g_3 = g_1 g_2$. The transforms g_k, $k=1,2,3$, do not alter the analytical properties of the Jost functions.

Define this group $\mathcal{G}$ as an automorphism group that acts on the set of fundamental solutions $\psi(\chi, \tau; \zeta)$ of Eqs. (34) and (35):

$$g: \psi(\chi, \tau; \xi) \to \hat{U}(g)\psi(\chi, \tau; g(\xi)) \in \{\psi(\chi, \tau; \xi)\}. \tag{47}$$

This group-element action is described by the transforms $\hat{U}(g_1)$, $\hat{U}(g_2)$, and $\hat{U}(g_3) = \hat{U}(g_1)\hat{U}(g_2)$.

The symmetry properties of the matrices $\mathbf{L}(\xi)$, $\mathbf{L}(\xi)$, and $\mathbf{F}_\pm(\xi)$ are

$$\mathbf{L}(-\xi) = \hat{\sigma}_3 \mathbf{L}(\xi)\hat{\sigma}_3, \quad \mathbf{A}(-\xi) = \hat{\sigma}_3 \mathbf{A}(\xi)\hat{\sigma}_3,$$

$$\mathbf{F}_\pm(-\xi) = \hat{U}(g_2)^{-1}\mathbf{F}_\pm(\xi) \equiv \hat{\sigma}_3 \mathbf{F}_\pm(\xi)\hat{\sigma}_3, \tag{48}$$

$$\mathbf{L}(q^2/\bar{\xi}) = \hat{\sigma}_1 \overline{\mathbf{L}(\xi)}\hat{\sigma}_1, \quad \mathbf{A}(q^2/\bar{\xi}) = \hat{\sigma}_1 \overline{\mathbf{A}(\xi)}\hat{\sigma}_1,$$

$$\mathbf{F}_\pm(q^2/\bar{\xi}) = \hat{U}(g_1)^{-1}\mathbf{F}_\pm(\xi) \equiv \frac{\bar{\xi}}{iq}\overline{\mathbf{F}_\pm(\xi)}\hat{\sigma}_1. \tag{49}$$

The symmetry transforms $\xi \to \bar{\xi}$ and $\xi \to q/\xi$ relate the Jost functions with different analytical properties. For example, the first transform acts as follows:

$$\overline{\mathbf{L}(\bar{\xi})} = \hat{\sigma}_1 \mathbf{L}(\xi)\hat{\sigma}_1, \quad \overline{\mathbf{A}(\bar{\xi})} = \hat{\sigma}_1 \mathbf{A}(\xi)\hat{\sigma}_1,$$

$$\overline{\mathbf{F}_\pm(\bar{\xi})} = \hat{\sigma}_3 \mathbf{F}_\pm(\xi)\hat{\sigma}_3. \tag{50}$$

Taking into account these symmetry properties we find that the transforms of the scattering coefficients under the action of the elements of the substitution group S are

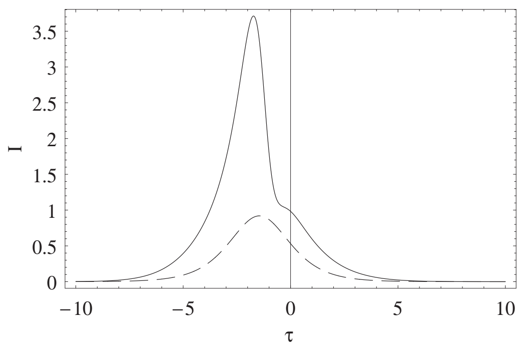

FIG. 1. Dependence of $I=\mathcal{E}_x^2+\mathcal{E}_y^2$ on τ for $\xi_1=0.5(1+i)$. The solid line corresponds to $m_x=0.6\mu_y^2$, and the dashed line corresponds to $m_x=0$.

$$a(\chi;g_1(\xi))=\overline{a(\chi;\xi)},\quad b(\chi;g_1(\xi))=\overline{b(\chi;\xi)},\tag{51}$$

$$a(\chi;g_2(\xi))=a(\chi;\xi),\quad b(\chi;g_2(\xi))=-b(\chi;\xi),\tag{52}$$

$$a(\chi;g_3(\xi))=\overline{a(\chi;\xi)},\quad b(\chi;g_3(\xi))=-\overline{b(\chi;\xi)}.\tag{53}$$

The symmetry properties of the coefficient $c(\chi;\xi_k)=b(\chi;\xi_k)/\partial_\xi a(\chi;\xi)|_{\xi=\xi_k}$, where ξ_k is the simple zero of $a(\chi;\lambda)$, are the following:

$$c(\chi;g_1(\xi_k))=-\frac{q^2}{\xi_k^2}\overline{c(\chi;\xi_k)},\tag{54}$$

$$c(\chi;g_2(\xi_k))=c(\chi;\xi_k),\tag{55}$$

$$c(\chi;g_3(\xi_k))=-\frac{q^2}{\xi_k^2}\overline{c(\chi;\xi_k)}.\tag{56}$$

Let the zeros ξ_{0k}, $k=1,2,\ldots,n$, of $a(\chi;\xi)$ be not degenerate—i.e., $|\xi_{0k}|\neq q^2,0,\infty$. It follows from the symmetry (46) that the zeros

$$\xi_{0k},\xi_{1k}=\frac{q^2}{\overline{\xi}_{0k}},\quad \xi_{2k}=-\xi_{0k},\quad \xi_{3k}=-\frac{q^2}{\overline{\xi}_{0k}}\tag{57}$$

are equivalent points, Fig. 1; see also [20].

Introduce the matrix-valued functions $\mathbf{M}(\tau;\xi)$ and $\mathbf{N}(\tau;\xi)$:

$$\mathbf{M}:=(\phi\, e^{(i\alpha_- -i\alpha_0)\sigma_3+i\Lambda\tau},\ \tilde{\phi}\, e^{(-i\alpha_- +i\alpha_0)\sigma_3-i\Lambda\tau}),\tag{58}$$

$$\mathbf{N}:=(\tilde{\psi}\, e^{-i\alpha_+\sigma_3+i\Lambda\tau},\ \psi\, e^{i\alpha_+\sigma_3-i\Lambda\tau}).\tag{59}$$

Then Eq. (34), for instance, for the first column $\mathbf{M}_1=(M_{11},M_{12})^T$ of the matrix $\mathbf{M}$ is

$$\partial_\tau M_{11}(\tau)=-i[\lambda(\xi)^2-\Lambda(\xi)+\partial_\tau\alpha_-(\tau)]M_{11}(\tau)+\lambda(\xi)E\, e^{2i\alpha_-(\tau)-2i\alpha_0}M_{12}(\tau),\tag{60}$$

$$\partial_\tau[M_{12}(\tau)e^{-2i\Lambda\tau}]=i[\lambda(\xi)^2-\Lambda(\xi)+\partial_\tau\alpha_-(\tau)]M_{11}(\tau)e^{-2i\Lambda\tau}-\lambda(\xi)\bar{E}e^{-2i\alpha_-(\tau)+2i\alpha_0}M_{12}(\tau)e^{-2i\Lambda\tau}.\tag{61}$$

The transforms (58) and (59) correspond to the following transforms of asymptotics:

$$\mathbf{F}_-(\tau;\xi)\to\tilde{\mathbf{F}}_-(\xi)=\begin{pmatrix}1 & -iq\xi^{-1}e^{-2i\alpha_0}\\ -iq\xi^{-1}e^{2i\alpha_0} & 1\end{pmatrix},\quad \tau\to-\infty,\tag{62}$$

$$\mathbf{F}_+(\tau;\xi)\to\tilde{\mathbf{F}}_+(\xi)=\begin{pmatrix}1 & -iq\xi^{-1}\\ -iq\xi^{-1} & 1\end{pmatrix},\quad \tau\to\infty.\tag{63}$$

Integrating Eqs. (60) and (61) yields

$$M_{11}(\tau)=1-\int_{-\infty}^{\tau}\{i[\lambda(\xi)^2-\Lambda(\xi)+\partial_y\alpha_-(y)]M_{11}(y)-\lambda(\xi)E\, e^{2i\alpha_-(y)-2i\alpha_0}M_{12}(y)\}dy,\tag{64}$$

$$\partial_\tau[M_{12}(\tau)e^{-2i\Lambda(\xi)\tau}]=\int_{-\infty}^{\tau}\{i[\lambda(\xi)^2-\Lambda(\xi)+\partial_y\alpha_-(y)]M_{12}(y)e^{-2i\Lambda(\xi)y}-\lambda(\xi)\bar{E}e^{-2i\alpha_-(\tau)+2i\alpha_0}M_{11}(y)e^{-2i\Lambda(\xi)y}\}dy.\tag{65}$$

By solving Eqs. (64) and (65) and analogous equations for the second column $\mathbf{N}_2$ of the matrix function $\mathbf{N}$ we found that the asymptotics

$$M_{11}(\tau;\xi)=1,\quad M_{11}(\tau;\xi)=0,\quad |\xi|\to\infty,\tag{66}$$

$$N_{21}(\tau;\xi)=0,\quad N_{22}(\tau;\xi)=1,\quad |\xi|\to\infty,\tag{67}$$

following from (62) and (63), as well as the symmetry properties of the Jost functions, yield the relations

$$\alpha_\pm(\tau,\chi)=\frac{1}{2}\int_{\pm\infty}^{\tau}[q^2-|E(\tau',\chi)|^2]d\tau',\tag{68}$$

$$\alpha_0=\alpha_-(\tau,\chi)-\alpha_+(\tau,\chi)=\frac{1}{2}\int_{-\infty}^{\infty}[q^2-|E(\tau',\chi)|^2]d\tau'.\tag{69}$$

Note that the linear system (64) and (65) possesses the same symmetry properties as the system (34).

Starting from the solutions (66) and (67) and iterating Eqs. (64) and (65) we find for large $|\xi|$

$$\mathbf{M}_1(\tau;\xi)=\begin{pmatrix}1\\ -i\xi^{-1}\bar{E}\, e^{2i\alpha_- -2i\alpha_0}\end{pmatrix}+O\left(\frac{1}{|\xi|^2}\right).\tag{70}$$

Analogously, we derive

$$\mathbf{N}_2(\tau;\xi)=\begin{pmatrix}-i\xi^{-1}E\, e^{-2i\alpha_+}\\ 1\end{pmatrix}+O\left(\frac{1}{|\xi|^2}\right).\tag{71}$$

By using the transform g_1 to Eqs. (70) and (71) one is able to derive the corresponding limits at $|\xi|\to 0$.

V. RIEMANN-HILBERT PROBLEM FORMULATION

Let us combine a pair of the following functions:

$$\Psi_+(\tau,\chi;\xi)=\left(\frac{\mathbf{M}_1(\tau,\chi;\xi)}{a(\chi;\xi)},\mathbf{N}_2(\tau,\chi;\xi)\right),$$

$$\Psi_-(\tau,\chi;\xi)=\left(\mathbf{N}_1(\tau,\chi;\xi),\frac{\overline{\mathbf{M}_2(\tau,\chi;\xi)}}{\overline{a(\chi,\bar{\xi})}}\right), \tag{72}$$

having canonical normalization at $|\xi|\to\infty$,

$$\lim_{|\xi|\to\infty}\Psi_\pm(\tau,\chi;\xi)=\mathbf{I}, \tag{73}$$

where $\mathbf{I}$ is the unit 2×2 matrix. Ψ_+ is analytical in D^+:{Im $\xi^2>0$}, and Ψ_- is analytical in D^-:{Im $\xi^2<0$}.

The Riemann-Hilbert problem (RHP) is formulated in the following manner:

$$\Psi_+(\tau,\chi;\xi)=\Psi_-(\tau,\chi;\xi)\mathbf{J}_+(\tau,\chi;\xi),\quad \mathrm{Im}\,\xi=0, \tag{74}$$

$$\Psi_+(\tau,\chi;\xi)=\Psi_-(\tau,\chi;\xi)\mathbf{J}_-(\tau,\chi;\xi),\quad \mathrm{Re}\,\xi=0. \tag{75}$$

Here $\Psi=\Psi_+$ for $\xi\in D^+$:{Im $\xi^2>0$} and $\Psi=\Psi_-$ for $\xi\in D^-$:{Im $\xi^2<0$}, and

$$\mathbf{J}_\pm(\tau,\chi;\xi)=\begin{pmatrix}1\pm\rho(\chi;\xi)\tilde{\rho}(\chi;\xi) & \pm\tilde{\rho}(\chi;\xi)e^{-2i\Lambda(\xi)\tau}\\ \rho(\chi;\xi)e^{2i\Lambda(\xi)\tau} & 1\end{pmatrix}, \tag{76}$$

$$\rho(\chi;\xi)=\frac{b(\chi;\xi)}{a(\chi;\xi)},\quad \tilde{\rho}(\chi;\xi)=\frac{\overline{b(\chi;\bar{\xi})}}{\overline{a(\chi;\bar{\xi})}}. \tag{77}$$

The solution of the RHP, taking into account analytical continuation into the respective regions of the complex plane ξ, is

$$\Psi_1^-(\chi,\tau;\xi)=\begin{pmatrix}1\\ -iq\xi^{-1}\end{pmatrix}+\frac{1}{2\pi i}\times\int_\Gamma\rho(\chi;\zeta)e^{2i\Lambda(\zeta)\tau}\Psi_2^+(\chi,\tau;\zeta)\frac{d\zeta}{\zeta-\xi}. \tag{78}$$

Here $\Psi_k^\pm$ is the kth column of the function $\Psi^\pm$. The contour Γ runs over the boundary of the region Im $\xi^2>0$ in a counterclockwise direction. The soliton and breather-type solution is associated with a finite number of zeros of $a(\chi,\xi_{mk})=0$, $m=0,1,2,3$, $k=1,2\ldots:\xi_{mk}\in D^+$. Deriving the first term on the RHS of Eq. (78) we took into account the symmetry property (49) related asymptotics $|\xi|\to\infty$ and $|\xi|\to 0$. We need to take it into account because the constructed solution $\Psi^\pm$, must be automorphic with respect to the action of group $\hat{G}$.

To derive a relation between the "potential" E and solution of the RHP we take the limit $\xi\to\infty$ in Eqs. (77) and (78) and compare the coefficients before $1/\xi$. We derive

$$\overline{E(\chi,\tau)}e^{2i\alpha_-(\chi,\tau)-2i\alpha_0}=q+\frac{1}{2\pi}\int_\Gamma\rho(\chi;\zeta)\Psi_{22}^+(\chi,\tau;\zeta)e^{2i\Lambda(\zeta)\tau}d\zeta. \tag{79}$$

Here $\Psi_{22}^+(\chi,\tau;\zeta)$ is the component of the matrix function $\Psi^+(\chi,\tau;\zeta)$.

We derive the dependence of the scattering coefficients on χ in a standard manner from Eq. (35) for $S_x(0,\chi)=S_y(0,\chi)=0$:

$$b(\chi;\xi)=b(0;\xi)e^{-2iR_z(0,\chi)\Delta(\xi)\chi}, \tag{80}$$

$$c(\chi;\xi_{0n})=c(0;\xi_{0n})e^{-2iR_z(0,\chi)\Delta(\xi_{0n})\chi}, \tag{81}$$

$$a(\xi)=a(0),\quad \xi(\chi)=\xi(0). \tag{82}$$

Here

$$\Delta(\xi)=\frac{\Lambda(\xi)}{\lambda(\xi)^2-1}. \tag{83}$$

Let the zeros $\xi_{0k}=\xi_k$ lie in the fundamental domain $D_1^+=\{|\xi|>|q|\cap\mathrm{Im}\,\xi>0\cap\mathrm{Re}\,\xi>0\}$. Considering the contribution of the discrete spectrum (57), we find from (79)

$$E(\chi,\tau)=q\,e^{2i\alpha_-(\chi,\tau)-2i\alpha_0}-2e^{2i\alpha_-(\chi,\tau)-2i\alpha_0}\times\sum_{n=1}^N\left[\frac{qc_n}{\xi_n}\phi_{1n}e^{2i\Lambda(\xi_n)\tau}+i\bar{c}_n\overline{\phi_{2n}}e^{-2i\overline{\Lambda(\xi_n)}\tau}\right]. \tag{84}$$

Here $\phi_{kn}=\Psi_{2k}^+(\chi,\tau;\xi_n)$, $k=1,2$, and $c_n=c(0;\xi_n)$.

VI. SIMPLEST NONTRIVIAL SOLUTION

Let us derive the simplest nontrivial breather-type solution, corresponding to the poles (57) for $k=1$ only. Then Eq. (78), taking into account the properties (54)–(56), is

$$\tilde{\Phi}_2(\chi,\xi)=\begin{pmatrix}1\\ -iq\xi^{-1}\end{pmatrix}+\frac{2c_1}{\xi^2-\xi_1^2}\begin{pmatrix}\xi_1 & 0\\ 0 & \xi\end{pmatrix}\Psi_2^+(\tau,\xi_1)e^{2i\Lambda_1\tau}+\frac{2iq\bar{c}_1}{\xi^2\bar{\xi}_1^2-q^4}\begin{pmatrix}0 & q^2\\ \xi\bar{\xi}_1 & 0\end{pmatrix}\overline{\Psi_2^+(\tau,\xi_1)}e^{-2i\bar{\Lambda}_1\tau}. \tag{85}$$

Here $\Lambda_1=\Lambda(\xi_1)$, $\tilde{\Phi}_{21}(\xi)=\overline{\Phi_{22}^+(\bar{\xi})}$, and $\tilde{\Phi}_{22}(\xi)=-\overline{\Phi_{12}^+(\bar{\xi})}$. To obtain self-consistent equations we put $\xi=\bar{\xi}_1$ on both sides of (85).

Let introduce the real constants γ, β, τ_0, and τ_1 such that

$$\xi_1=qe^{\gamma+i\beta},\quad c_1(0)=qe^{\gamma_1+i\beta_1}, \tag{86}$$

$$c_1(\chi)e^{2i\Lambda_1\tau}=-iq\omega_1\sin(2\beta)e^{\gamma+i\beta+\theta+i\vartheta}. \tag{87}$$

Here

$$\omega_1=\frac{1}{2}(e^{2\gamma+2i\beta}-e^{-2\gamma-2i\beta}), \tag{88}$$

$$\theta=-q^2\cosh(2\gamma)\sin(2\beta)(\tau-\chi v^{-1}-\tau_0), \tag{89}$$

$$\vartheta = q^2 \sinh(2\gamma)\cos(2\beta)(\tau - \chi u^{-1} - \tau_1), \tag{90}$$

$$\gamma_1 = q^2 \cosh(2\gamma)\sin(2\beta)\tau_0, \tag{91}$$

$$\beta_1 = -q^2 \sinh(2\gamma)\cos(2\beta)\tau_1, \tag{92}$$

$$v^{-1} = \frac{1}{w}\left[\frac{m_x^2}{2}\frac{\cosh(2\gamma)}{\cos(2\beta)} - \mu_x\mu_y\right], \tag{93}$$

$$u^{-1} = \frac{1}{w}\left[\frac{m_x^2}{2}\frac{\cos(2\beta)}{\cosh(2\gamma)} - \mu_x\mu_y\right], \tag{94}$$

$$w = \frac{q^4}{4}[\cos(2\beta)^2 + \sinh(2\gamma)^2] - q^2\cosh(2\gamma)\cos(2\beta) + 1.$$

The solution to (84) and (85) corresponding to the initial condition $S_z(0,\chi)\equiv -1$ is

$$E(\chi,\tau) = q\, e^{2i\alpha_-(\chi,\tau)-2i\alpha_0}\frac{N}{Z}, \tag{95}$$

where

$$N = 1 + \sin(2\beta)e^{\theta-3i\beta}(e^{3\gamma-i\vartheta} - e^{i\vartheta-3\gamma}) + \sinh^2(2\gamma)e^{2\theta-6i\beta}, \tag{96}$$

$$Z = 1 + \sin(2\beta)e^{i\beta+\theta}(e^{-\gamma-i\vartheta} - e^{i\vartheta+\gamma}) + \sinh^2(2\gamma)e^{2\theta+2i\beta}. \tag{97}$$

It can be directly verified that

$$q^2 - |E|^2 = 2\left(\frac{\partial_\tau Z}{iZ} - \frac{\partial_\tau \bar{Z}}{i\bar{Z}}\right). \tag{98}$$

Thus

$$i\alpha_-(\tau,\chi) = \ln\frac{Z}{\bar{Z}} - 4\beta. \tag{99}$$

From (99) and the asymptotics $E\to q$ as $\tau\to\pm\infty$ we obtain

$$\alpha_0 = -4\beta. \tag{100}$$

With relations (37), (95), and (99) we derive

$$\mathcal{E}_x - i\mathcal{E}_y = \frac{m_x}{\mu_y^2}\frac{\bar{Z}^2 - ZN}{\bar{Z}^2}. \tag{101}$$

VII. CONCLUSIONS AND APPLICATIONS

Self-induced transparency is described here for a two-level system interacting with a circularly polarized pulse with duration nearly equal to, or greater than, the inverse transition frequency. Analysis of behavior of solutions to the RMB equations is important for understanding the UDP formation and self-induced transparency of few-cycle pulses in two-level systems. Formally, the RMB equations considered here are applicable in the spectral interval between $\tau_s^{-1} \ll \omega_0$ and $\tau_s^{-1} \gg \omega_0$ where τ_s is the soliton duration. However, the condition $\tau_s^{-1} \lesssim \omega_0$ must be imposed on optical pulses. The solutions obtained in this study demonstrate that the dynamics of circularly polarized UDPs is qualitatively different from that of linearly polarized UDPs and from the behavior of 2π pulses in the McCall-Hahn theory [3]. For example, the McCall-Hahn theorem is not applicable to UDPs with durations comparable to the inverse transition frequency. Therefore, the results of the self-induced transparency theory developed for quasimonochromatic pulses cannot be extended to pulses with $\tau_s^{-1} \sim \omega_0$, especially to circularly polarized pulses, for which nonlinear effects due to the interactions between field components are essential. It is found that circularly polarized electromagnetic pulses can be used to obtain pulses of higher intensity, as compared to the case of linearly polarized pulses of equal duration. Accordingly, circularly polarized pulses have shorter durations as compared to linearly polarized ones with equal peak amplitudes.

Analysis of the solution (101) shows that the PDM ($m_y \neq 0$) in the case of two-component EMF may lead to compression of propagating pulses and enlargement of its amplitude in comparison to the case of zero PDM. For illustration the form of the intensity of the EMF described by (101) is depicted in Fig. 1 for zero and nonzero PDMs for the same ξ_1 and time shift. The solution (101) demonstrates qualitative effects based on the polarization properties and the PDM in comparison with known solutions. For example, the form of the pulse propagated in an anisotropic medium depends on an initial polarization of the EMF. These properties may be used to control the pulse parameters and sensing of the medium structure.

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